hep-th/9611234

Derivation of quantum theories: symmetries and the exact solution of the derived system

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Abstract

Based on the technique of derivation of a theory, presented in our recent paper [1], we investigate the properties of the derived quantum system. We show that the derived quantum system possesses the (nonanomalous) symmetries of the original one, and prove that the exact Green functions of the derived theory are expressed in terms of the semiclassically approximated Green functions of the original theory.

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1 Introduction

In recent decades some two-dimensional exactly solvable quantum field theories have been discovered. 2-dimentinall conformal field theory [2] and 2-dimensional Yang-Mills theory [3] are two important examples of them. There are also some recent advances in $d > 2$ dimensions, where the 4-dimensional supersymmetric gauge theories is one of the most important ones [4].

In this paper we want to discuss a class of exactly solvable quantum theories. In the paper [1], it was shown that, starting from an action, one can construct other actions which the classical behaviour of them are deduced from those of the original action. The procedure relating the original action to the new one is just a derivation. Now we want to continue our investigation by studying the behaviour of a quantum system obtained by derivation from an action of a quantum theory. It turns out that the derived system possesses all of the (nonanomalous) symmetries of the original system, just as the case of classical systems. Here, however, a novel property arises: the derived quantum theory is *almost classical*; that is in the derived theory there are only one loop quantum corrections to the classical action. Using this property, one can calculate all of the Green functions of the derived theory exactly, even though this may be not the case for the original theory. Note that the classical equations of motion are the same in the original theory and the derived one, although their quantum behaviour are completely different.

The paper consists of three sections. In section 1, we show that the derived quantum system possesses the (nonanomalous) symmetries of the original theory. In section 2, we study a specific example: a Coulomb gas theory, which has conformal symmetry. We show that the derived theory is a combination of two theories : a Coulomb gas theory which its central charge is related to the derivative of that of the first and a free bosonic theory. Finally, in section 3 we prove that the generating functional of the connected diagrams of the derived theory has only tree- and one loop-contributions, and these have simple relations to the tree- and one loop-parts of the generating functional of the connected diagrams of the original theory. So, solving the original theory up to one loop allows one to solve the derived theory exactly.

2 Action formulation, preservation of symmetries

Consider the action

$$S^{(0)} = S(\phi, \lambda), \tag{1}$$

where ϕ denotes the dynamical variable(s) of the system and λ is some parameter. As in [1], differentiating this action with respect to λ, treating ϕ as a function of λ, leads to:

$$S^{(1)} := \frac{dS^{(0)}}{d\lambda} = \frac{\partial S}{\partial \lambda} + \Big\langle \frac{\delta S}{\delta \phi}, \psi \Big\rangle, \tag{2}$$

where we have defined

$$\psi := \frac{d\phi}{d\lambda}. \tag{3}$$

In [1], it was shown that if the first action has a symmetry, the derived action $S^{(1)}$, possesses this symmetry as well. We now want to show that if the symmetry of the first theory is not anomalous, the symmetry of the derived one is not anomalous either.

To do this, one must show that the path integral measure corresponding to the derived theory does not change under the symmetry transformation. We know that, under the symmetry transformation

$$\phi_\lambda \to \mathcal{O}_\lambda \phi_\lambda, \tag{4}$$

the measure does not change:

$$D\phi_\lambda = D(\mathcal{O}_\lambda \phi_\lambda), \tag{5}$$

since the symmetry of the first theory has no anomaly. This shows that the symmetry of the theory defined through the action

$$S^{(1)\Delta} := \frac{1}{\Delta}\Big[S(\phi_{\lambda+\Delta}, \lambda+\Delta) - S(\phi_\lambda, \lambda)\Big], \tag{6}$$

is nonanomalous. Now consider the following change of variables

$$\begin{aligned} \psi^\Delta &:= \frac{1}{\Delta}(\phi_{\lambda+\Delta} - \phi_\lambda) \\ \phi &:= \phi_\lambda. \end{aligned} \tag{7}$$

The Jacobian of this transformation is a constant:

$$D\psi^\Delta \; D\phi = \Big(\frac{1}{\Delta}\Big)^N D\phi_{\lambda+\Delta} \; D\phi_\lambda. \tag{8}$$

This shows that, under the transformation

$$\begin{aligned} \phi &\to \mathcal{O}_\lambda \phi \\ \psi^\Delta &\to \frac{1}{\Delta}[\mathcal{O}_{\lambda+\Delta}(\phi + \Delta\psi^\Delta) - \mathcal{O}_\lambda \phi], \end{aligned} \tag{9}$$

the measure corresponding to ϕ and ψ^Δ does not change:

$$D\psi^\Delta \; D\phi = D\Big\{\frac{1}{\Delta}[\mathcal{O}_{\lambda+\Delta}(\phi + \Delta\psi^\Delta) - \mathcal{O}_\lambda \phi]\Big\} D(\mathcal{O}_\lambda \phi). \tag{10}$$

Now let Δ tends to zero. The transformation (9) becomes

$$\begin{aligned} \phi &\to \mathcal{O}_\lambda \phi =: \mathcal{O}\phi \\ \psi &\to \frac{d}{d\lambda}(\mathcal{O}_\lambda \phi) = \Big(\frac{\partial \mathcal{O}_\lambda}{\partial \lambda}\phi + \frac{\partial}{\partial \phi}(\mathcal{O}_\lambda \phi)\psi\Big) =: \mathcal{O}\psi, \end{aligned} \tag{11}$$

and therefore

$$D\phi \; D\psi = D(\mathcal{O}\phi) \; D(\mathcal{O}\psi). \tag{12}$$

This completes the proof.

3 A simple example: derivation of a Coulomb gas theory

As a simple example, consider a Coulomb gas theory with the following energy–momentum tensor [5]

$$T^{(0)}(z) = -\frac{1}{2} : \partial\phi\partial\phi + \frac{Q}{2}\partial^2\phi. \tag{13}$$

It is well known that this is a conformal field theory (CFT) with the central charge

$$c^{(0)} = 1 + 3Q^2, \tag{14}$$

and the following OPE of the fields

$$\phi(z)\phi(w) = -\ln(z-w) + \text{ regular terms}. \tag{15}$$

Differentiating (13) with respect to Q, we obtain

$$T^{(1)}(z) = - : \partial\phi\partial\psi : + \frac{Q}{2}\partial^2\psi + \frac{1}{2}\partial^2\phi. \tag{16}$$

As this component of the energy-momentum tensor is independent of $\bar{z}$, the derived theory is conformal as well. Now, if $T^{(1)}$ is to be the energy–momentum tensor of a new theory, which contains the fields ϕ and ψ, we must have the following OPE's

$$\begin{aligned} T^{(1)}(z)\phi(w) &= \frac{\partial\phi(w)}{z-w} + \text{ terms containing no derivative} + \text{ regular terms} \\ T^{(1)}(z)\psi(w) &= \frac{\partial\psi(w)}{z-w} + \text{ terms containing no derivative} + \text{ regular terms}. \end{aligned} \tag{17}$$

This relations are satisfied, provided $\phi(z)\phi(w)$ and $\psi(z)\psi(w)$ are regular and

$$\phi(z)\psi(w) = -\ln(z-w) + \text{ regular terms}, \tag{18}$$

that is, provided we have

$$< \phi(z)\phi(w) > = < \psi(z)\psi(w) > = 0, \tag{19}$$

and

$$< \phi(z)\psi(w) > = -\ln(z-w). \tag{20}$$

(19) and (20) are a special case of the correlation functions of a derived theory to be discussed in the next section.

Using the above-mentioned OPE's, it is readily seen that

$$T^{(1)}(z)T^{(1)}(w) = \frac{1+3Q}{(z-w)^4} + \frac{2T^{(1)}(w)}{(z-w)^2} + \frac{\partial T^{(1)}(w)}{z-w} + \text{ regular terms}, \tag{21}$$

which shows that $T^{(1)}$ is indeed the energy-momentum tensor of a CFT with the central charge

$$c^{(1)} = 2 + 6Q = 2 + \frac{dc^{(0)}}{dQ}. \tag{22}$$

It can be also shown that the fields

$$\begin{aligned} V_1 &:= \phi - Q\psi \\ V_2 &:= \partial V_1 \end{aligned} \tag{23}$$

are the primary fields of this theory with the conformal weights $h_1 = 0$ and $h_2 = 1$, respectively. Also, one can write $T^{(1)}$ in terms of two new fields

$$\begin{aligned} \chi_1 &:= \frac{1}{\sqrt{2Q}}(\phi - Q\psi) \\ \chi_2 &:= \frac{1}{\sqrt{2Q}}(\phi + Q\psi) \end{aligned} \tag{24}$$

as

$$T^{(1)}(z) = \frac{1}{2} : \partial\chi_2\partial\chi_2 - \frac{1}{2} : \partial\chi_2\partial\chi_2 + \frac{Q'}{2}\partial^2\chi_2. \tag{25}$$

where

$$Q' := \sqrt{2Q}. \tag{26}$$

The fields χ_1 and χ_2 satisfy the following OPE's:

$$\begin{aligned} \chi_1(z)\chi_1(w) &= \ln(z-w) + \text{ regular terms}, \\ \chi_2(z)\chi_2(w) &= -\ln(z-w) + \text{ regular terms}, \\ \chi_1(z)\chi_2(w) &= \text{ regular terms}. \end{aligned} \tag{27}$$

So χ_1 and χ_2 are two independent fields, and the central charge of the derived theory is the sum of two central charges: the central charge of a free bosonic theory ($c = 1$), and that of a Coulomb gas theory ($c = 1 + 3Q'^2 = 1 + 6Q$), which is the same as the result obtained earlier. One can continue the derivation. After n times derivation, we will have

$$T^{(n)} = \frac{d^n T}{dQ^n}, \tag{28}$$

which is the energy momentum tensor of a CFT with central charge

$$c^{(n)} = n + 1 + \frac{d^n c^{(0)}}{dQ^n}. \tag{29}$$

4 Exact form of the generating functional of a derived action

Consider an action $S^{(0)}(\phi)$ of the form (1), which contains no first order terms in ϕ, and its derived action $S^{(1)}(\phi,\psi)$, which is of the form (2). The generating functional corresponding to such an action is

$$Z^{(1)}(J,j) := \int D\phi\, D\psi\, \exp\left[\frac{i}{\hbar}\left(< \frac{\delta S^{(0)}}{\delta\phi}, \psi > + \frac{\partial S^{(0)}}{\partial\lambda} - < j,\psi > - < J,\phi >\right)\right]. \tag{30}$$

Integrating over ψ, and then ϕ, we obtain

$$\begin{aligned} Z^{(1)}(J,j) &:= \int D\phi\, \delta\left(\frac{\delta S^{(0)}}{\delta\phi} - j\right) \exp\left[\frac{i}{\hbar}\left(\frac{\partial S^{(0)}}{\partial\lambda} - < J,\phi >\right)\right] \\ &= \left\{\left[\det\left(\frac{\delta^2 S^{(0)}}{\delta\phi^2}\right)\right]^{-1} \exp\left[\frac{i}{\hbar}\left(\frac{\partial S^{(0)}}{\partial\lambda} - < J,\phi >\right)\right]\right\}|_{\phi=\phi(j)}, \end{aligned} \tag{31}$$

where $\phi(j)$ is the solution of the classical equation of motion of ϕ with the source j:

$$\frac{\delta S^{(0)}}{\delta\phi}|_{\phi(j)} = j. \tag{32}$$

From $Z^{(1)}$, one obtains the generating functional of connected diagrams:

$$W^{(1)} := \ln Z^{(1)} \qquad \Rightarrow \tag{33}$$

$$\begin{aligned} W^{(1)}(J,j) &= -\frac{i}{\hbar} < J,\phi(j) > + \frac{i}{\hbar}\frac{\partial S^{(0)}(\phi(j))}{\partial\lambda} - \mathrm{tr}\ \ln\left[\frac{\delta^2 S^{(0)}}{\delta\phi^2(j)}\right] \\ &=: W^{(1)}_{(0)}(J,j) + W^{(1)}_{(1)}(J,j), \end{aligned} \tag{34}$$

where the subscripts refer to the number of loops. So the exact generating functional has only zero- and one-loop contributions. We can compare this to the generating functional of the original theory, up to the first loop order. We have

$$Z^{(0)}(j) := \int D\phi\ \exp\left[\frac{i}{\hbar}\left(S^{(0)} - < j,\phi >\right)\right]. \tag{35}$$

Expanding ϕ around $\phi(j)$, and keeping only terms up to second order, we obtain

$$\begin{aligned} Z^{(0)}(j) &:= \int D\phi\ \exp\left\{\frac{i}{\hbar}\left[S^{(0)}[\phi(j)] + \frac{1}{2} < \phi-\phi(j), \frac{\delta^2 S^{(0)}}{\delta\phi^2}|_{\phi(j)}[\phi-\phi(j)] > - < j,\phi(j) >\right]\right\} \\ &= \exp\left\{\frac{i}{\hbar}\left[S^{(0)}[\phi(j)] - < j,\phi(j) >\right]\right\}\left[\det\left(\frac{\delta^2 S^{(0)}}{\delta\phi^2(j)}\right)\right]^{-1/2}, \end{aligned} \tag{36}$$

and from this,

$$\begin{aligned} W^{(0)}_1(j) &= \frac{i}{\hbar}\left\{S^{(0)}[\phi(j)] - < j,\phi(j) >\right\} - \frac{1}{2}\mathrm{tr}\ \ln\left[\frac{\delta^2 S^{(0)}}{\delta\phi^2(j)}\right] \\ &=: W^{(0)}_{(0)}(j) + W^{(0)}_{(1)}(j). \end{aligned} \tag{37}$$

Comparing this with (34), we see that

$$W^{(1)}_{(0)}(J,j) = < J, \frac{\delta W^{(0)}_{(0)}(j)}{\delta j} > + \frac{dW^{(0)}_{(0)}(j)}{d\lambda}, \tag{38}$$

and

$$W^{(1)}_{(1)}(J,j) = W^{(1)}_{(1)}(j) = 2W^{(0)}_{(1)}(j). \tag{39}$$

Note that (37) is an approximation of the generating functional of the original theory, whereas (34) is the exact generating functional of the derived one.

From (38), it is easily seen that

$$\frac{\delta^2 W^{(1)}}{\delta J^2} = 0, \tag{40}$$

which means that

$$< \phi_1 \phi_2 \cdots >_c^{(1)} = 0, \tag{41}$$

where the dots denote any combination of ϕ's and ψ's, and the superscript refers to the derived theory. We also have

$$i\hbar \frac{\delta W^{(1)}}{\delta J} = i\hbar \frac{\delta W^{(0)}_{(0)}}{\delta j}. \tag{42}$$

So

$$(i\hbar)^{n+1} \frac{\delta^{n+1} W^{(1)}}{\delta j^n \delta J} = (i\hbar)^{n+1} \frac{\delta^{n+1} W^{(0)}_{(0)}}{\delta j^{n+1}}, \tag{43}$$

which means

$$< \phi_0 \psi_1 \cdots \psi_n >_c^{(1)} = < \phi_0 \cdots \phi_n >_{c,(0)}^{(0)}; \tag{44}$$

that is, the exact connected Green function of one ϕ and n ψ's is equal to the Green function of $n+1$ ϕ's in the original theory at the tree level.

Finally,

$$(i\hbar)^n \frac{\delta^n W^{(1)}}{\delta j^n}\Big|_{J=0} = 2(i\hbar)^n \frac{\delta^n W^{(0)}_{(1)}}{\delta j^n} + (i\hbar)^n \frac{d}{d\lambda} \frac{\delta^n W^{(0)}_{(0)}}{\delta j^n}, \tag{45}$$

which means

$$< \psi_1 \cdots \psi_n >_c^{(1)} = 2 < \phi_1 \cdots \phi_n >_{c,(1)}^{(0)} + \frac{d}{d\lambda} < \phi_0 \cdots \phi_n >_{c,(0)}^{(0)} . \tag{46}$$

To summerize, we showed that one can calculate all of the exact connected Green functions of the derived theory in terms of the connected Green functions of the original theory up to one loop.

There exists another proof for this result. Consider the transformation (7) and the definition (6). It is seen that

$$\begin{aligned} < \phi_0 \cdots \phi_k \psi^\Delta_{k+1} \cdots \psi^\Delta_n >_c^{(1)\Delta} &= \frac{(-1)^{n-k}}{\Delta^{n-k}} < \phi_0 \cdots \phi_n >_c^{(1)\Delta} \\ &= \frac{(-1)^{n-k}}{\Delta^{n-k}} < \phi_0 \cdots \phi_n >_{c,-\Delta\hbar}^{(0)}, \end{aligned} \tag{47}$$

where the last relation means the $n+1$ point function calculated using $-\Delta\hbar$ instead of $\hbar$. Now we have

$$< \phi_0 \cdots \phi_n >_{c,-\Delta\hbar}^{(0)} = < \phi_0 \cdots \phi_n >_{c,(0),-\Delta\hbar}^{(0)} + < \phi_0 \cdots \phi_n >_{c,(1),-\Delta\hbar}^{(0)} + \cdots, \tag{48}$$

which is an expansion in terms of powers of $-\Delta\hbar$. The zeroth term is of the order $(-\Delta\hbar)^n$. It is seen that if $k > 0$, the limit of the right-hand side of (47) at $\Delta \to 0$ is zero, and if $k = 0$, only the zeroth term of the right-hand side of (48) survives in the limit, which yields (44).

To calculate the n point function of ψ's, we note that

$$< \psi_1 \cdots \psi_n >_c^{(1)\Delta} = \frac{1}{\Delta^n} \left[< \phi_1 \cdots \phi_n >_{c,\Delta\hbar}^{(0)} |_{\lambda+\Delta} + (-1)^n < \phi_1 \cdots \phi_n >_{c,-\Delta\hbar}^{(0)} |_{\lambda} \right]$$

$$= \frac{1}{\Delta^n} \Big[< \phi_1 \cdots \phi_n >_{c,(0),\Delta\hbar}^{(0)} |_{\lambda+\Delta} + < \phi_1 \cdots \phi_n >_{c,(1),\Delta\hbar}^{(0)} |_{\lambda+\Delta} + \cdots$$

$$+ (-1)^n < \phi_1 \cdots \phi_n >_{c,(0),-\Delta\hbar}^{(0)} |_{\lambda} + (-1)^n < \phi_1 \cdots \phi_n >_{c,(1),-\Delta\hbar}^{(0)} |_{\lambda} + \cdots \Big] . \tag{49}$$

The first term in $< \phi_1 \cdots \phi_n >_{c,\Delta\hbar}^{(0)}$ is of order $(\Delta\hbar)^{n-1}$. So

$$< \psi_1 \cdots \psi_n >_c^{(1)} = \lim_{\Delta \to 0} \frac{< \phi_1 \cdots \phi_n >_{c,(0)}^{(0)} |_{\lambda+\Delta} - < \phi_1 \cdots \phi_n >_{c,(0)}^{(0)} |_{\lambda}}{\Delta} + 2 < \phi_1 \cdots \phi_n >_{c,(1)}^{(0)}, \tag{50}$$

which is precisely (46).

One can contninue the procedure and differentiate further. In a similar manner, it turns out that only zero- and one-loop terms contribute to the Green functions.

Acknowledgement M. Khorrami and M. Alimohammadi would like to thank the research vice-chancellor of the university of Tehran, the work was partially supported by them.

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